Non-Commutative Methods in Quantum Mechanics

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Abstract

This is the text of a presentation based on the Princeton University Physics Department doctoral thesis of the same name.

What does "Non-Commutative Methods in Quantum Mechanics" mean? After all, anyone familiar with quantum mechanics knows that operators (or, equivalently, matrices) do not necessarily commute, as a consequence of which there are uncertainty relations placing limits on the accuracy of measurements of observables:

$$\Delta P \, \Delta Q \; \geq \; \frac{1}{2} \, | \, \langle \, i \, [\, \check{P}, \; \check{Q} \,] \, \rangle \, | \; .$$

However, while the operators (or matrices) may not commute, the usual formulation of quantum mechanics takes matrix elements and wavefunction expansion coefficients to be complex numbers, all of which commute with one another. One of the ideas behind my thesis, and the work of Stephen Adler, is that it is worth investigating formulations of quantum mechanics where not only are the operators non-commuting, but so are the matrix elements and expansion coefficients. I think that there are then three questions that need to be answered in order to defend this programme of research:

1. What is the motivation?
2. What experimental predictions are made?
3. How does it fit into existing physics?

In the course of this presentation, I shall attempt to answer these questions for the specific case of the material presented in my thesis, using material from my thesis to illustrate my answers.

First, what is the motivation? Why take this step to generalise the algebra of matrix elements and expansion coefficients? Well, let me make, probably unnecessarily, two points:

1. High energy physics (that is, that part of physics that deals with fundamental particles and interactions) doesn't have all of the answers.
2. Physicists don't know where to look for the answers.

This is putting the situation rather bluntly, so let me justify these statements. At the moment, we have a pretty good description of the strong, electroweak and gravitational interactions at low energies in terms of local, complex quantum field theories, but we believe that that's not the whole story. Primarily, we believe that there is a theory that unifies these interactions at some higher energy. We believe that such a theory also explains why there are three generations of fermions, as well as predicts the Cabibbo–Kobayashi–Maskawa matrix elements. We believe that such a theory would also yield the particle masses (or the strengths of their Higgs couplings) and resolve the hierarchy problem. However, as my second blunt statement says, we're having problems finding this theory. Of course, there are some clues: supersymmetry, for instance, seems pretty natural, but that's not the same as knowing where to look. (Table I from "Unification and Supersymmetry" by Mohaptra presents some of the attempts that have been made to reach towards a unifying theory.)

Of course, this is a rather general answer to the question, but it sets the scene and justifies investigating new ideas. In the case of generalising the algebra of matrix elements and expansion coefficients, though, we are taking a second, closer look at material we think we already understand. As Steve writes in his introduction to "Quaternionic Quantum Mechanics and Quantum Fields",

> we expect to get a better understanding of a system of postulates if we have more than one concrete realisation. Specifically, we can expect to gain a deeper understanding of standard, complex quantum mechanics if we understand which features of the usual formalism are more general than others.

And we have indeed achieved a better and deeper understanding of some parts of quantum mechanics. For example, chapter 2 of my thesis, "Quantum Mechanics as a Classical Linear System", focuses on the postulate that quantum mechanics is a specialisation of classical mechanics, which may readily be demonstrated for complex quantum mechanics as follows. Schrödinger's equation is

$$i\frac{d}{dt}|\Psi> = \check{H}|\Psi> \ ,$$

where $\check{H}$ is a self-adjoint Hamiltonian operator. Expand the wavefunction over an orthonormal basis,

$$|\Psi> = \sum_r |\phi_r>\psi_r \ ,$$

and write the expansion coefficients in terms of their real and imaginary parts,

$$\psi_r = (q_r + ip_r)/\sqrt{2} \ .$$

In terms of the q_r and the p_r and the real quantity

$$H = <\Psi|\check{H}|\Psi> \ ,$$

Schrödinger's equation can be written

$$i(\dot{q}_r + i\dot{p}_r)/\sqrt{2} = \frac{1}{\sqrt{2}}\left(\frac{\partial H}{\partial q_r} + i\frac{\partial H}{\partial p_r}\right)$$

or, separating into real and imaginary parts,

$$\dot{q}_r = \frac{\partial H}{\partial p_r} \ , \ \dot{p}_r = -\frac{\partial H}{\partial q_r} \ ,$$

which are the classical dynamical equations for a phase space with canonical coordinates q_r and p_r and a Hamiltonian function H.

Now, this has been known for a while. The above demonstration, which I think shows the correspondence between complex quantum mechanics and classical dynamics with exceptional clarity, was given by Heslot in the paper "Quantum Mechanics as a Classical Theory" in 1985 and I find it incredible that it's not common knowledge. (I'll be saying that again, about something else, in a moment.)

So in generalising quantum mechanics to retain this correspondence, what do we learn?

1. Associativity is crucial. Steve shows, in his book, that quantum mechanics cannot be formulated over the octonions since, in at least two instances, associativity is needed for the existence of Hilbert space — similar arguments can be extended to other non-associative algebras.

This, I suppose, is a technical point, though I understand that there have been some attempts to use octonions, for instance in the so-called exceptional Jordon algebra and contemporaneously, so I'm told, in string theory.

2. Being able to convert between self-adjoint and anti-self-adjoint operators is an artifact of complex quantum mechanics since, in general, there is no central (totally commuting) unit anti-self-adjoint operator like the $i\check{1}$ of complex quantum mechanics.

This means that the generators of unitary transformations must be left as anti-self-adjoint generators instead of factoring out an i to give self-adjoint operators. Hence, Schrödinger's equation must be written

$$\frac{d}{dt}|\Psi> = -\check{H}|\Psi> \ ,$$

where the Hamiltonian operator $\check{H}$ is now anti-self-adjoint; similarly, the generators of $SU(2)$ should satisfy

$$[\check{S}_i, \check{S}_j] = \sum_k \epsilon_{ijk}\check{S}_k \ .$$

I'll return to $SU(2)$ shortly, with something about quaternionic coherent states.

3. There are some correspondences between features of quantum mechanics and features of classical mechanics.

Quantum Mechanics	Classical Dynamics
real-valued parameters of expansion coefficients	phase space coordinates
anti-self-adjoint operators generating transformations	"admissable" functions generating phase flows
the commutator	the poisson bracket

Now look at the third correspondence. We're all told, in a more-or-less hand-waving manner, that the commutator in quantum mechanics plays the rôle of the Poisson bracket in classical dynamics. Even Dirac, in "The Principles of Quantum Mechanics", can only provide what amounts to a plausibility argument. However, the correspondence is exact (for anti-self-adjoint operators, at least):

$$\langle\, [\, \check{F},\ \check{G}\,]\, \rangle \;=\; \{\, \langle\, \check{F}\, \rangle\, ,\, \langle\, \check{G}\, \rangle\, \}\ .$$

This is where I repeat myself and say that I don't understand why this is not common knowledge, even for complex quantum mechanics.

Now chapter 2 of my thesis stresses, in a couple of places, that its considerations do not include the issue of measurement. If expansion coefficients are to be probability amplitudes, then there is a further restriction to division algebras. Furthermore, since Nature seems to favour real numbers, allowing us to redefine our units of measurement by rescaling by any real number we choose, we need to consider associative division algebras over the reals. There are only three of them, and they are the reals, the complex numbers and the quaternions.

The problem with quantum mechanics over the reals is that there are no states that are stationary under a transformation (energy eigenstates, for example) except for those that are annihilated by the transformation (zero energy). This is a consequence of quantum mechanics being a specialisation of classical dynamics: in a classical phase flow, the phase space coordinates pair up or, as Arnold puts it in "Mathematical Methods of Classical Physics",

> an odd-dimensional manifold cannot admit a symplectic structure.

However, a stationary quantum state essentially means that there is a sub-manifold under the transformation, where the dimension of the sub-manifold is the number of real-valued parameters in a state's expansion coefficient. For quantum mechanics over the reals, this is just one, which is of course odd, so there is no symplectic structure. For quantum mechanics over the complex numbers and over the quaternions, the numbers are two and four respectively, so this argument cannot be used against them. We will, however, reinstate the reals as a viable possibility later, since they are an acceptable algebra for use in Adler's trace dynamics.

Quantum mechanics over the complex numbers is very successful, as we all know. However, it has only been in recent years that quantum mechanics over the quaternions has been considered, with the most notable watershed being Steve's book. With a few exceptions, quantum mechanics can be carried over almost unchanged from the complex numbers to the quaternions. One of the most interesting features from the point of view of experimental verification of quaternionic quantum mechanics is that the asymptotic states of a scattering process lie in a complex sub-space of the quaternionic Hilbert space. In other words, the S-matrix is complex yet, in general, not time reversal invariant. This means that, even if there are quaternionic interactions, experiments such as that of Kaiser, George and Werner, reported in 1984, are expected to yield a null result. In their experiment, titanium and aluminium slabs were inserted in one arm of a neutron interferometer, with each slab thick enough to produce a large phase shift. They measured the difference in the phase shifts between the two orderings of the slabs and found that

$$| \delta_{\mathrm{Al-Ti}} \ - \ \delta_{\mathrm{Ti-Al}} | \leq 0.3^{\circ} \ .$$

This is expected from Steve's analysis, since scattering phase shifts belong to the same complex sub-algebra, and therefore commute. On the other hand, violation of time reversal symmetry could be indicative of quaternionic physics. Neutron experiments to detect such a violation (inspired by the possibility of a $\mathcal{CP}$-violating phase in the CKM matrix) have not so far found anything.

As mentioned earlier, quantum mechanics over a general algebra does not permit the trivial interconversion of self-adjoint and anti-self-adjoint operators, and this remains true for quaternionic quantum mechanics. This means that the eigenvalues of the (anti-self-adjoint) Hamiltonian, corresponding to energy, and of the (anti-self-adjoint) translation operators, corresponding to momenta, are, first, imaginary and, second, only unique up to a rotation of the imaginaries. This rotation corresponds to a redefinition or "reraying" of our chosen eigenstates. Now, if these were (anti-self-adjoint) operators in complex quantum mechanics, we'd need some sort of "stability condition" (as yielded, for example, by a Dirac sea of negative energy states) as an additional postulate to guarantee that energy is bounded from below. However, in quaternionic quantum mechanics, a basis of states can always be found such that the Hamiltonian and translation operators (being conserved) are diagonal, with all the eigenvalues using the same unit imaginary, and with one of the eigenvalues restricted to a positive coefficient of that imaginary. With the already existing requirement of manifest spatial isotropy, it makes sense to choose this distinguished eigenvalue (and its corresponding operator) as the energy (and Hamiltonian). Hence, there is no need for an extra "stability condition". Moreover, the point of zero energy is significant since, in quantum mechanics over the quaternions, it cannot be freely shifted. In the same way, Nature appears to regard the zero point of the vacuum energy as significant, given that the cosmological constant vanishes, and hence quaternionic quantum mechanics can provide an explanation of this significance, which is lacking in complex quantum mechanics.

Another difference between quantum mechanics over the quaternions and over the complex numbers arises in the consideration of coherent states, with which chapter 5 of my thesis, "Quaternionic Coherent States" deals. As I said before, much of complex quantum mechanics carries over to quaternionic quantum mechanics, and the same is true of Perelomov's coherent states formalism. For a given symmetry group and initial fixed state, we can construct a set of coherent states with the usual phase relationships for transforming from one state to another. The set of coherent states is overcomplete, though with appropriate normalisation, an identity operator can be defined in terms of them, and an arbitrary state may be expanded over coherent states. Apart from the form of the phase relationships, which differ because phase arguments cannot simply be added if they do not commute, the differences between the quaternionic and complex coherent states formalisms arise when specific symmetry groups are considered.

For instance, the nilpotent or Weyl, group is generated by a set of annihilation $\{\check{a}_i\}$ and creation $\{\check{a}_i^\dagger\}$ operators, where $i = 1, \ldots, N$, with commutation relations

$$[\check{a}_i, \check{a}_j] = [\check{a}_i^\dagger, \check{a}_j^\dagger] = [\check{a}_i, 1] = [\check{a}_i^\dagger, 1] = 0$$

and

$$[\check{a}_i, \check{a}_j^\dagger] = \delta_{ij} 1 \ .$$

Anti-self-adjoint generators of transformations can then be constructed in terms of these operators, and coherent states defined. However, closure of the group imposes such restrictions that the various coefficients of the operators may only range over a complex sub-algebra of the quaternions. Hence, the coherent states are not fully quaternionic. On the other hand, there are fully quaternionic coherent states for the odd half-integer representations of $SU(2)$. I gave the anti-self-adjoint generators of $SU(2)$ earlier as

$$[\check{S}_i, \check{S}_j] = \sum_k \epsilon_{ijk} \check{S}_k$$

and it's easy to see that these are basically the commutation relations for the half unit imaginaries of the quaternions, that is with:

$$\check{S}_1 = \frac{1}{2} i \ , \ \check{S}_2 = \frac{1}{2} j \ , \ \check{S}_3 = \frac{1}{2} k \ .$$

This, then, is a one-dimensional representation of $SU(2)$, something that just doesn't exist in complex quantum mechanics, and is a representation for spin $\frac{1}{2}$. Choosing the fixed state as the $+\frac{1}{2}$ eigenstate of $\check{S}_3$,

$$|\frac{1}{2}> = 1 \quad \Rightarrow \quad \check{S}_3 \ |\frac{1}{2}> = |\frac{1}{2}> \ \frac{1}{2} k \ ,$$

the coherent states may be characterised by a unit vector (in a three dimensional space) or, equivalently, by an azimuthal angle and a polar angle,

$$|\hat{\mathbf{n}}\rangle = |\theta, \phi\rangle = \exp \frac{1}{2}\phi k \exp \frac{1}{2}\theta j \ .$$

The identity operator may be given as

$$\check{1} = \frac{1}{4\pi} \int |\hat{\mathbf{n}}\rangle\langle\hat{\mathbf{n}}| \ d\hat{\mathbf{n}} \ .$$

There are no even half-integer (that is, whole integer) quaternionic representations of $SU(2)$, but there are other odd half-integer representations, as calculated, for instance, by Finkelstein, Jauch and Speiser. These too have quaternionic coherent states, which are n-component quaternionic spinors for the spin $\frac{1}{2}(2n-1)$ representation.

Now John Klauder pointed out that the $N = 1$ nilpotent group is a limiting case of $SU(2)$, so he asked how this fits with the fact that the nilpotent group has no fully quaternionic coherent states, yet $SU(2)$ does. Here's Klauder's construction. Form, using the usual self-adjoint operators of complex quantum mechanics $\check{S}_x$, $\check{S}_y$ and $\check{S}_z$, the usual ladder operators so that

$$[\check{S}_+, \check{S}_-] = \check{S}_z \ , \ [\check{S}_z, \check{S}_\pm] = \pm\check{S}_\pm \ .$$

Thus,

$$\left[\frac{\check{S}_+}{\sqrt{S}}, \frac{\check{S}_-}{\sqrt{S}}\right] = \frac{\check{S}_z}{S} \ , \ \left[\frac{\check{S}_\pm}{\sqrt{S}}, \frac{\check{S}_z}{S}\right] = \mp\frac{\check{S}_\pm}{S\sqrt{S}} \ .$$

Given a state with $M \approx S \gg 1$,

$$\check{S}_z|M\rangle = |M\rangle M \approx |M\rangle S \ ,$$
$$\check{S}_\pm|M\rangle = |M \pm 1\rangle\frac{1}{\sqrt{2}}\sqrt{(S \mp M)(S \pm M + 1)} \approx |M \pm 1\rangle\sqrt{S},$$

and so

$$\lim_{S \to \infty} \frac{\check{S}_z}{S} = \check{1}$$
$$\lim_{S \to \infty} \frac{\check{S}_+}{\sqrt{S}} = \check{a}$$
$$\lim_{S \to \infty} \frac{\check{S}_-}{\sqrt{S}} = \check{a}^\dagger \ ,$$

with these operators possessing the same commutation relations as the $N = 1$ nilpotent group. Now, this construction was performed using the usual operators of complex quantum mechanics, and the divergence with quaternionic quantum mechanics occurs at the point of formation of the ladder operators. In the construction, these operators are

$$\check{S}_{\pm} = \frac{1}{\sqrt{2}}(\check{S}_x \pm i\check{S}_y) ,$$

where we have used complex quantum mechanics' freedom to make an anti-self-adjoint operator, $i\check{S}_y$, from a self-adjoint operator, $\check{S}_y$. In quaternionic quantum mechanics, we'd want to go the other way, to make two self-adjoint operators $\check{S}_x$ and $\check{S}_z$ that are the equivalents of our original anti-self-adjoint $\check{S}_1$ and $\check{S}_3$. However, we can't do this. We could make one operator by finding a unit anti-self-adjoint operator $\check{I}$ that commutes with $\check{S}_3$ and forming $\check{S}_z = \check{I}\check{S}_3$, but then that same $\check{I}$ will not commute with $\check{S}_1$ and so cannot be used to make the required $\check{S}_x$. On the other hand, using a different unit anti-self-adjoint operator for the construction of $\check{S}_x$ would give incorrect commutation relations for $\check{S}_x$ and $\check{S}_z$. Hence, Klauder's construction breaks down in quaternionic quantum mechanics, and there is no conflict with having quaternionic representations for $SU(2)$ but not for the nilpotent group.

The final difference that I will consider concerns projective representations, as discussed in chapter 4 of my thesis, "Quaternionic Projective Representations". A central feature of wavefunctions is that we expect to be able to rephase them,

$$|\psi\rangle \mapsto |\psi\rangle\omega$$

where $\omega^*\omega = 1$, and not change the physical content of the state. If the algebra of these phases is the reals, $\omega = \pm 1$; if it is the complex numbers, $\omega = \exp i\theta$; if it is the quaternions,

$$\omega = \cos\theta_1 + i\sin\theta_1\cos\theta_2 + j\sin\theta_1\sin\theta_2\cos\theta_3 + k\sin\theta_1\sin\theta_2\sin\theta_3 ,$$

where

$$0 \leq \theta_1 < 2\pi \ , \ 0 \leq \theta_2 < \pi \ , \ 0 \leq \theta_3 < \pi \ .$$

Projective representations arise because, if two group elements combine to give a third,

$$a \circ b = ab ,$$

the corresponding multiplication need not be genuine or faithful,

$$\check{U}_a\check{U}_b = \check{U}_{ab} \ ,$$

but could involve a phase,

$$\check{U}_a\check{U}_b|\psi\rangle = \check{U}_{ab}|\psi\rangle\omega(a,b;\psi) \ .$$

Now in complex quantum mechanics, this is taken to hold for all states, and it then leads to the usual theory of projective representations. However, in the case of quaternionic quantum mechanics, it forces the projective phases to be ± 1, as a function of a and b only, for all states, so the only representations here are real projective representations. This is therefore called the strong definition, and is clearly too restrictive to be useful. In contrast, the weak definition assumes that the projective relation holds only for one particular complete set of states, $\{|\phi_i\rangle\}$. In the case of complex quantum mechanics, again, this leads to the strong definition anyway, but for quaternionic quantum mechanics, representations other than real projective representations now exist. For the case of a connected group, and making extensive use of restrictions on the projective phases that arise because of the requirement of associativity,

$$(\check{U}_a\check{U}_b)\check{U}_c = \check{U}_a(\check{U}_b\check{U}_c) \ ,$$

there is then a structure theorem, according to which there are three possibilities, namely that the representation is either real projective, complex projective or is the product of real projective and a quaternionic phase. The proof of the theorem consists of considering the different possibilities for the ranges of the projective phases and how they commute as a set.

We have seen that quaternionic quantum mechanics has many of the features of complex quantum mechanics, yet also exhibits new features and may well be able to deal with issues, such as the vanishing of the cosmological constant, which are extremely difficult to explain within the framework of complex quantum mechanics. And this is really only the tip of the iceberg: the material of chapters 4 and 5 of my thesis, on quaternionic projective representations and quaternionic coherent states, addresses or answers questions 38 and 16 posed in the last section of Steve's book, out of a total of 46 numbered questions.

Of course, we know that quantum mechanics in the form of the one-particle Schrödinger (or Dirac) equation won't be the machinery with which a unifying theory is constructed: we need field theory for that. This brings me to chapter 3 of my thesis, "Trace Dynamics", which discusses some aspects of trace dynamics, a generalised Lagrangian formalism where the phase space variables are operators acting on a state space. Although Steve's motivation for developing trace dynamics was to handle quaternionic field theories, trace dynamics is quite suited to handling matrix elements and expansion coefficients from the complex numbers and even the reals, and the state space may have both a bosonic and a fermionic sector, giving a system that is admirably suited to performing calculations in supersymmetric models, as Steve is now investigating. Trace dynamics is in fact a classical symplectic dynamics, with a Poisson bracket and the equations of motion given in terms of a Hamiltonian. It is, however, non-local and so although it gives a "hidden variables" interpretation for the probabilistic structure of quantum measurement, the locality assumption that is the basis for Bell's inequalities is not valid.

My contribution to the lore of trace dynamics concerns an operator that is the difference between the sum of the commutators of the bosonic coordinates and the sum of the anti-commutators of the fermionic coordinates,

$$\check{C} \; = \; \sum_{\substack{i=1 \\ \text{bosonic}}}^{N} [\check{q}_i, \check{p}_i] \; - \; \sum_{\substack{i=1 \\ \text{fermionic}}}^{N} \{\check{q}_i, \check{p}_i\} \; .$$

This operator, which is anti-self-adjoint, is a constant of the motion so long as the operator Hamiltonian satisfies certain restrictions — being Weyl-ordered or constructed from monomials formed from the coordinates using only central coefficients is sufficient, so that there is global unitarity. Since the trace of the operator Hamiltonian is also conserved, these two constants allow statistical mechanics to be applied to an equilibrium ensemble of initial values for the operator phase space, the ensemble being characterised by Lagrange multipliers that correspond to the conserved quantities. Assuming that the dynamics is ergodic, ordinary complex quantum field theory emerges as a statistical approximation to the underlying trace dynamics, whether that was over the reals, the complex numbers or the quaternions, with

$$< \; \check{C} \; > = \; i\hbar \; .$$

To round off, let me make some comments concerning the promise that I believe is shown by trace dynamics. First, the actual, underlying trace dynamics need not be unitary, offering a resolution to the paradoxes of quantum measurement, namely Schrödinger's cat, Wigner's friend and the rest of the von Neumann chain. One avenue of speculation is that for a large enough system, such as one satisfying Penrose's "one graviton criterion", be it the Planck mass at $10^{-5}g$ or the GUT mass at $10^{-9}g$, asymptotic scattering states will not all belong to the same complex sub-algebra and/or will not evolve with unitarity. Hence, the quaternionic and/or non-unitary effects will be compounded in large, multiparticle systems. One particularly interesting comment that Steve makes is that

> there may be a connection between the large ratio of scales characterising the hierarchy problem and the very high accuracy to which conventional linear quantum mechanics is observed to hold.

Or, to put it another way, linear and/or complex quantum mechanics breaks down (that is, the wavefunction collapses) when we attempt to bridge the gap between a microscopic, Higgs scale system and a macroscopic Planck–GUT scale system, which is when we try to make a measurement. (It is interesting to note that amoebae exist at masses as large as one hundred times the Planck mass and as small as one hundredth of the GUT mass. Other speculative possibilities include having criteria based on a length scale or on some measure of complexity of the system.)

My second comment concerns the involvement of gravitation. Formulating trace dynamics over the reals, complex numbers or quaternions corresponds to working with matrices having elements from these algebras, due to the usual correspondence between operators and matrix elements. Now these matrices, in particular those of dimension 2^n, are the lowest dimensional representations of the possible algebras constructed from sets of gamma matrices, as used in the Dirac equation at $n = 2$ and in supersymmetric models with somewhat larger n. These matrices anti-commute with one another,

$$\{\gamma_i, \gamma_j\} = \eta_{ij} \ ,$$

where, usually, $\eta_{ij} = \pm\delta_{ij}$. Most commonly, the gamma matrices employed by theorists are complex, as in Wilczek's and Zee's construction in "Families from Spinors", yet only a quarter of the gamma matrix algebras are complex. Of the remaining algebras, half are over the reals and half are over the quaternions. (Table II in chapter 4 of "Spin Geometry" by Lawson and Michelsohn illustrates the diversity of representations for some low dimensional algebras.) However, while the real gamma matrices have been used in a few places, I know of no places in the wider high energy physics forum that uses these quaternionic representations, so people may well be ignoring from $\frac{3}{8}$ to $\frac{3}{4}$ of the possible theories before they even begin. Moreover, there are correspondences between the gamma matrices and basis vectors: it's no accident that for the Dirac gamma matrices,

$$\{\gamma_\mu, \gamma_\nu\} = g_{\mu\nu} \ ,$$

where the quantity on the right is the metric for ordinary space-time. In particular, if the gamma matrices depend on position, then their connections are the usual Christoffel quantities for the space (or space-time). From these, the Riemann curvature tensor can be constructed, and hence some part of the machinery of general relativity can be invoked. Hence, trace dynamics could conceivably provide a framework for a unified description of gravitation and quantum mechanics.

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Further references may be found via the bibliographies of my thesis and of Steve's book.